IV. The Calculus

Several years ago, a group of the brightest minds of our time got together to discuss the greatest achievements of mankind. Their choice for the number one achievement was not the wheel, relativity, or computers. It was the calculus. In this chapter, we will discuss the achievement, the men behind it, and why it is so important.

Section 1: Ancient Roots

There are many different calculi: calculus of variations, differential calculus, vector calculus, and many others. In fact the word "calculus" is not just a math term but an English word as well:

> Random House: calˈ cu lus *n*. A method of calculation by a special system of algebraic notations.

So what is the calculus and why does it stand out in the history of mathematics? Calculus as we know it is essentially broken into two halves: differential calculus and integral calculus. It is interesting to note that while modern-day calculus classes begin with differentiation and proceed to integration, historically integration developed first. The fundamental notions of both branches – derivatives and integrals – involve the paradoxical idea of infinity and infinitesimals. While the majority of this chapter takes place in 17^{th} century Europe, we begin in 5^{th} century BC Greece.

Are magnitudes infinitely divisible? Or are there smallest parts that can be divided no further? In other words, you can take one half of something, and then take one half of that, and so on. But do you reach a point at which you can no longer subdivide the magnitude? This was the question the ancient Greeks struggled with. Intuitively, it seems that we should be able to infinitely divide a magnitude (like a distance or time), but the Greek philosopher Zeno (c. 450 BC) described a couple of paradoxes that challenged this notion. In "Achilles," Zeno explains that in a race, a slower runner given a head start will never be caught by a faster runner, even Achilles. To overtake the slower runner, Achilles must first run to the point the slower runner started. In the time it takes him to do this, the slower runner will have run some distance. Achilles must then run to THAT point, yet the slower runner will again have made progress. This can be continued indefinitely. Another version of this paradox (called "The Dichotomy") illustrates that in order to move a certain distance, you must first move half that distance. But before you can do that, you must first move a fourth that total distance. Prior to this, you must first move one-eighth the total distance and so on forever. So clearly, you can never move. Both these paradoxes "show" that distances must not be infinitely divisible.

However, if magnitudes are made up of indivisible "atomic" particles, then we have the trouble described in "The Arrow". Here Zeno explains that if time is made up of atomic "instants" that cannot be subdivided, then at any one of them a moving arrow must be at rest. Therefore, the arrow can never move. The result of Zeno's paradoxes was unfortunately that infinitesimals were not allowed in Greek geometry and arithmetic. But they did take a peek into the world of the infinite. One of the classic problems of antiquity was the *squaring the circle* problem. The goal here was to construct (using only a straightedge and a compass) a square whose area was the same as that of a given circle. Today, we know this problem to be unsolvable, but the ancient Greeks tried to solve it by inscribing a polygon within the circle and repeatedly doubling the number of sides. They reasoned that eventually the polygon would be identical with the circle. Even though this lacks a good bit of rigor and reason, it was the first attempt to deal with an infinite process.

This brings us to one of the two motivating problems of calculus: calculating the area (or volume) of a given region. Archimedes generalized the above process and explained that if the given region was cut into a very large number of very thin parallel strips, they could be rearranged and formed into a polygon and the area would be calculable. He was unable to make this method fully rigorous (in particular he lacked the notion of a limit), and was unable to prove his method worked, but this is essentially the same as our modern method of integration.

The other problem that motivated the development of the calculus was the slope of the tangent problem. For this discussion, we jump ahead in time about 2000 years.

Section 2: Fermat and Descartes

One of the most important pre-calculus developments was the invention of analytic geometry by Rene Descartes and Pierre Fermat. The marriage of algebra and geometry is a wonderful development in the history of mathematics. By the 17^{th} century, geometry was one of the oldest and most well-respected fields of mathematics. Despite this, there were many problems that were still proving to be too difficult to handle geometrically. Along came algebra. As we have seen, in the preceding centuries algebra had been advancing at a tremendous pace due to the work of the Italian number theorists and the improved algebraic symbolism. Once algebra gained wide acceptance, it took the genius of Descartes and Fermat to understand that it could be used to address problems in geometry.

Rene Descartes (1596-1650) was one of the most influential thinkers in the history of mankind. In fact, he only published one work on mathematics. He was also a philosopher and scientist. Around 1633, Descartes was working on a book about the nature of the universe when he heard of the fate of Galileo. (Recall that The Church had condemned Galileo for advancing a theory that the Church viewed as contradictory to the Bible.) Descartes abandoned his work, and started working on *A Discourse on the Method of Rightly Conducting the Reason and Seeking Truth in the Sciences*. This philosophical treatise explained Descartes' views on science in general. It contained three appendices, the third of which was entitled *La geometrie*, and it was in this appendix that Descartes outlined his version of analytic geometry.

In *La geometrie*, Descartes described his coordinate system, which we now call the Cartesian coordinate system in his honor. His idea that you could identify every point in the plane with two coordinates was revolutionary. It allowed problems of geometry to be translated into problems involving variables that the "new" algebra could easily handle. There is an interesting anecdote about how Descartes came up with this idea. Legend has it that he was watching a fly crawling on his ceiling. He realized that he could describe the position of the fly by noting how far it was from each of the two nearby walls. Whether this is true or not, this new ability to relate a group of points to an algebraic equation was a powerful tool in advancing both algebra and geometry.

Pierre Fermat (1601-1665) was born in France to wealthy and hard working parents. His father was a leather merchant and his mother came from a family of jurists. Fermat followed his mother's family line, and pursued a degree in the civil laws, graduating from the University of Orleans in 1631. He became a counselor (lawyer) after graduating, and because of the high mortality rate of his colleagues, he advanced rapidly. Within 11 years, Fermat rose to the highest office in Toulouse, his hometown. From an early age, Fermat showed a great deal of interest classical works in many fields such as Latin, philosophy, literature, and mathematics. In the latter field, Fermat made extensive study of the ancient works of Archimedes, Apollonius, Euclid, Pappas, and especially Diophantus. However, in the early 17th century, mathematics was not a profession with employment opportunities outside of private tutoring. So while working as a lawyer, Fermat spent his free time on many mathematical problems and topics that would have enormous impact.

Fermat developed analytic geometry around the same time as Descartes, but from the opposite point of view. Whereas Descartes started with points and attempted to construct curves geometrically and then derive their equations, Fermat would start with the equation and derive the graph. It is this approach that more closely resembles our current method. Indeed the differing approaches soon caused a controversy. After receiving a copy of *La geometrie* sent by fellow Frenchman Jean Beugrand, Fermat paid it little attention. Finally, a friend and colleague Marin Mersenne asked Fermat to render an opinion of Descartes' work. Fermat said the Descartes was "groping about in the shadows." Thus began a feud between two great mathematicians.

It was Fermat however who first anticipated differentiation. In 1629, he described how to find maximum and minimum values of a function. After noting that the increment of a function becomes smaller and smaller nearby a maximum or a minimum (this was also observed earlier by Kepler), Fermat developed the following technique for finding those extreme values. Suppose $f(x)$ has a maximum value at x, and if e is very small, then the value of $f(x-e)$ will be almost equal to $f(x)$. Therefore, he tentatively set $f(x-e)=f(x)$, simplified the resulting equation (eventually setting $e=0$) and solved.

Example 1: What two numbers add to M (and arbitrary integer) and have a largest possible product?

Let the two numbers be A and $M-A$. Then we wish to maximize $f(A)=A(M-A)$. Consider $f(A-e)=(A-e)(M-(A-e))$. If we set this equal to $f(A)$ we get,

$$(A-e)(M-(A-e))=A(M-A) \text{, or}$$

$$2Ae-Me-e^2=0$$

Dividing through by e reduces this equation to $2A - M - e = 0$. If we then set $e = 0$, we see that the solution is $A = M/2$.

Obviously this method leaves a little to be desired, but you can see the hint of our modern method of setting the derivative equal to 0. Fermat also devised a method for finding the equation of a tangent line, again by employing the use of an arbitrarily small quantity e, which he then lets equal 0.

When Descartes read of Fermat's work on maxima, minima, and tangent lines, he was extremely critical. Descartes questioned the methods of Fermat. But in truth, he was mostly bothered by the fact that Fermat's work rendered much of *La geometrie* irrelevant. Once Fermat was able to verify his methods worked, Descartes wrote:

> ... seeing the last method that you use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would have not contradicted it at all.

We will have much more to say about Fermat in our next chapter (on number theory).

Section 3: Newton and Leibniz

So Archimedes, Descartes, and Fermat all made progress (in some cases significant progress) towards the calculus. But they are not considered the fathers of the field. That distinction belongs to Sir Isaac Newton and Gottfried Leibniz.

We have seen Newton's name already. In any discussion of the greatest mathematicians of all time, Newton is always a finalist. His brilliance showed through in many different fields, but he is chiefly remembered for inventing the calculus. Isaac Newton (1642-1727) was born the same year Galileo died. He was already an accomplished mathematician and scientist when he took over the Lucasian professorship at Cambridge from his colleague Isaac Barrow. It was at Cambridge that Newton made his revolutionary discoveries. From 1665 to 1667, Cambridge was closed due to the bubonic plague for all but about 3 months in the summer of 1666. It was during these three months that Newton developed his method of fluxions, what today we know as differential calculus.

Newton called any changing quantity a ***fluent*** and its rate of change was called the ***fluxion*** of the fluent. Using these terms and some interesting notations, Newton found the relationship between fluents and their fluxions (basically differentiation), maxima and minima, tangents, curvature, points of inflection, and concavity. He also made significant progress in differential equations, again using his notation.

Gottfried Leibniz (1646-1716) was a contemporary of Newton's, but they did not know each other personally. He invented his version of the calculus sometime between 1673 and 1676. Leibniz's notation was far superior to Newton's and in fact many symbols we use today are due to him, such as the $\int$ for the integral sign and the $dy/_{dx}$ notation. Additionally, many elementary rules for finding derivatives were derived by Leibnitz. In fact, the rule for finding the n^{th} derivative of the product of two functions is still called ***Leibniz' rule***.

A few words should be said about the controversy regarding who truly invented the calculus. We saw that Archimedes, Descartes, and Fermat (as well as many others) had invented or developed many of the ideas that would become calculus. What Newton and Leibniz did was unify them into a coherent theory and provide *the* system of rules for how to calculate things with these new tools. But who invented it first? Newton probably came up with his ideas prior to Leibniz by a few years, but he did not publish his findings until after Leibniz published his work. There is no evidence that Leibniz learned of Newton's work or that he used it to help develop his results. In fact, as we noted, his notations are significantly different, and better. Europe of the 18^{th} century viewed Newton as the inventor and Leibniz as a fraud, but that was largely due to Newton's reputation and fame. In the end, mathematicians have accepted that Newton AND Leibniz both invented the calculus.

Newton and Leibniz did not only invent the calculus though. Far from it, they were both remarkable men of great accomplishments. Leibniz taught himself Latin and Greek as a child and by the time he was 20 he was well versed in mathematics, philosophy, theology, and law. He developed the first ideas that would eventually become symbolic logic (some 200 years later). Newton is perhaps best known (aside from the calculus) for his work *Philosophiae naturalis principia mathematica* or the *Principia* as it is always known. It was in this epic work that he outlined a complete system of dynamics and a complete formulation of the principal terrestrial and celestial laws of motion. It made an enormous impression immediately throughout Europe and has become one of the most influential works of all time. Even Leibniz paid tribute to Newton by saying "Taking mathematics from the beginning of the world to the time when Newton lived, what he did was the better half." The poet Alexander Pope wrote:

Nature and Nature's laws lay hid in night;
God said 'Let Newton be', and all was light.

Newton himself was more humble, "If I have been able to see further, it was only because I stood on the shoulders of giants." Newton was known to have amazing abilities of concentration and focus. Several anecdotes exist, perhaps some not altogether true, demonstrating this to amusing ends. Here are two:

(a) On an occasion that a friend called Newton to a chicken dinner, Newton promptly forgot to go. After waiting some time, the friend removed the cover on the table and proceeded to eat the chicken, and then replaced the bones under the covered dish. When Newton finally recalled the invitation and later appeared, he greeted his friend, sat down at the table, lifted the cover, and discovered the remains. "Dear me," he said, "I had forgotten that we had already dined."

(b) Once riding home, he dismounted his horse in order to walk the animal up a steep hill. The horse slipped away on the way up the hill, but Newton did not discover this until, bridle in hand, he attempted to vault into a saddle that was no longer there.